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Non-conservative stability of multi-step non-uniform columns

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Abstract

The non-conservative stability of non-uniform columns under the combined action of concentrated and variably distributed forces is solved analytically. Two types of follower force system are considered: (i) concentrated follower forces and variably distributed follower forces, (ii) concentrated follower forces and variably distributed conservative forces. The exact solutions for stability of four kinds of one-step non-uniform columns subjected to the two types of follower force system are derived for the first time. Then a new exact approach, which combines the exact solutions of one-step columns and the transfer matrix method, is presented for the non-conservative stability analysis of multi-step non-uniform columns. The advantage of the proposed method is that the resulting eigenvalue equation for a multi-step non-uniform column with any kinds of two end support configurations, an arbitrary number of spring supports and concentrated masses can be conveniently determined from a second order determinant. The decrease in the determinant order, as compared with previously developed procedures, leads to significant savings in the computational effort. A numerical example shows that the results obtained from the proposed method are in good agreement with those determined from the finite element method (FEM), but the proposed method takes less computational time than FEM. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Non-conservative system; Stability; Buckling; Column; Exact solution; Transfer matrix method

1. Introduction

Lightweight structural members have been extensively used in many industrial fields such as in civil, mechanical and aerospace engineering, and therefore the stability problems of such structural members are of increasing importance. The applied loads are regarded as non-conservative forces if the work done by them is path-dependent. Practical examples of non-conservative forces include: (1) the aerodynamic drag forces acting on the body of rockets, missiles and other flight vehicles; (2) the forces acting on the rotor of a gas turbine; (3) the forces acting on the links and elements in automatic control systems (Bolotin, 1963). Meanwhile, a cantilever pipe conveying fluid is an example of a system subjected to follower forces (Sugiyama et al., 1999). Therefore, it is obvious that the concept of follower force is very important not only in

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aerospace engineering, but also in automobile engineering, and certainly also in the area of fluid-structures interaction technologies (Sugiyama et al., 1999). In fact, elastic systems subjected to non-conservative forces are always encountered in engineering practices. Thus, stability analysis of elastic structural members under non-conservative forces is important in modern engineering applications.

The non-conservative stability of an elastic uniform bar was first investigated by Nikolai (1928). Beck (1952) studied the stability of a uniform cantilever column subjected to a follower force at the free end. After Beck's work, many similar, but extended problems were investigated. For example, the stability problem of an elastically restrained-free Beck's column subjected to an end follower force was investigated by Simkins and Anderson (1975). Sundarajan (1976) studied the influence of an elastic end support on the stability of Beck's column. Pedersen (1977) examined the effect of a concentrated mass attached to the elastic support on the stability of Beck's column. Chen and Ku (1991) studied the stability of a uniform Timoshenko cantilever column subjected to follower force at the free end. McGill (1971) considered the stability problem of a uniform cantilever column under distributed vertical and follower forces using the Galerkin method. Sugiyama and Kawagoe (1975) studied the stability of elastic uniform columns with six typical boundary conditions subjected to combined action of uniformly distributed vertical and tangential forces by means of the finite difference method.

Much work has been done to investigate the stability of uniform structural members under follower forces, however, non-conservative stability of non-uniform columns received relatively less attention in the past. In fact, structural members with variable cross-section are frequently used in engineering practices to optimise the distributions of weight and strength. Massey and Van der Meen (1971) studied the stability of tapered cantilever columns subjected to a tangential tip load for breadth taper only. The effect of taper on the non-conservative stability of columns was studied by Sankaran and Rao (1976) using the finite element method (FEM). The stability problem of a non-uniform Timoshenko beam with clamped-free and elastically restrained-free boundary conditions subjected to three types of follower forces was solved by Irie et al. (1980). Lee and Kuo (1991) investigated the elastic stability of three different tapered columns subjected to uniformly distributed follower forces by dividing a non-uniform column into several uniform segments to simplify the stability analysis. It is revealed from the above cited references that the previous studies have generally concentrated their investigations on the stability of columns subjected to uniformly distributed follower forces. The stability problem of non-uniform columns, especially multi-step non-uniform columns, under variably distributed follower forces has rarely been studied. It is noted that the exact solution for such a problem has not been proposed in the past.

Literature survey indicates that the non-conservative stability of multi-step non-uniform columns was usually solved using numerical or approximate methods. In this paper, the non-conservative stability of non-uniform columns under the combined action of concentrated and variably distributed forces is solved analytically. Two types of follower force system are considered in this paper: (i) concentrated follower forces and variably distributed follower forces, (ii) concentrated follower forces and variably distributed conservative forces. The exact solutions for stability of four kinds of one-step non-uniform columns subjected to the two types of follower force system are derived for the first time. Then, a new exact approach, which combines the closed-form stability solutions for a one-step non-uniform column and the transfer matrix method, is proposed for the non-conservative stability analysis of multi-step non-uniform columns. The advantage of the proposed method is that the resulting eigenvalue equation for a multi-step non-uniform column with any kinds of two end support configurations, an arbitrary number of spring supports and concentrated masses can be conveniently determined from a second order determinant. As a consequence, the decrease in the determinant order, as compared with previously developed procedures, leads to significant savings in the computational effort. Numerical example demonstrates that the results obtained from the proposed method are in good agreement with those determined from FEM, but the proposed method takes less computational time than FEM, illustrating the present method is exact and efficient.

2. Theory

A multi-step non-uniform column under the combined action of concentrated follower forces at the end of each step and variably distributed follower forces along the column is shown in Fig. 1. The governing differential equation for stability of the ith step column can be written as

$$\frac{d^2}{dx^2}\left[K_i(x)\frac{d^2X_i(x)}{dx^2}\right] + N_i(x)\frac{d^2X_i(x)}{dx^2} - \overline{m}(x)\omega^2X_i(x) = 0 \tag{1}$$

where $X(x)$ is the mode shape function and ω is the circular natural frequency, $K(x)$, $N(x)$ and $\overline{m}(x)$ are the flexural stiffness, axial force and mass per unit length, respectively, the origin of the co-ordinate is set at the top end of the ith step column (Fig. 1).

If the variably distributed forces are the conservative loads, Eq. (1) should be replaced by

$$\frac{d^2}{dx^2}\left[K_i(x)\frac{d^2X_i(x)}{dx^2}\right] + \frac{d}{dx}\left[N_i(x)\frac{dX_i(x)}{dx}\right] - \overline{m}(x)\omega^2X_i(x) = 0 \tag{2}$$

In this paper, two types of follower force system acting on multi-step non-uniform columns are considered: (i) concentrated follower forces and variably distributed follower forces (type 1), (ii) concentrated follower forces and variably distributed conservative forces (type 2). The governing differential equations for the non-conservative stability of a non-uniform column subjected to the type 1 and type 2 kinds of follower force are Eqs. (1) and (2), respectively.

The general solutions of Eqs. (1) and (2) can be expressed in the following form

$$X_i(x) = C_{1i}S_{1i}(x) + C_{2i}S_{2i}(x) + C_{3i}S_{3i}(x) + C_{4i}S_{4i}(x) \tag{3}$$

where $S_{ji}(x)$ and C_{ji} $(j = 1, 2, 3, 4)$ are linearly independent solutions and integral constants of Eq. (1) or Eq. (2), respectively. Obviously, it is difficult to derive the closed-form solutions to Eq. (1) or Eq. (2) for general case, since $S_{ji}(x)$ are dependent on the expressions of $K_i(x)$, $N_i(x)$ and $\overline{m}_i(x)$. The analytical solutions may be obtained by means of reasonable selections for $K_i(x)$, $N_i(x)$ and $\overline{m}_i(x)$. As suggested by Li et al. (1994, 1998) and Li (2000), the functions for describing the variations of $K(x)$ and $\overline{m}(x)$ for many non-uniform structural members are power functions and exponential functions. Hence, the following four cases of $K_i(x)$, $N_i(x)$ and $\overline{m}_i(x)$ which cover many cases of non-uniform columns are considered.

Case 1. The distributions of flexural stiffness, axial force and mass per unit length of the ith step column are described by the following power functions

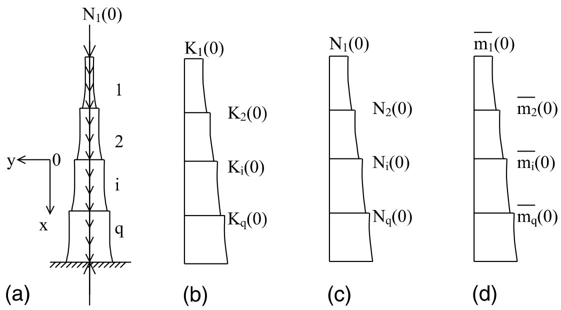

Fig. 1. A multi-step column.

$$\left.\begin{aligned} K_i(x) &= K_i(0)\left(1+\beta_i \tfrac{x}{L_i}\right)^{n+2} \\ N_i(x) &= N_i(0)\left(1+\beta_i \tfrac{x}{L_i}\right)^{n+1} \\ \overline{m}_i(x) &= \overline{m}_i(0)\left(1+\beta_i \tfrac{x}{L_i}\right)^{n} \end{aligned}\right\} \tag{4}$$

where $K_i(0)$, $N_i(0)$ and $\overline{m}_i(0)$ are the flexural stiffness, axial force and mass per unit length of the ith step column at $x = 0$, respectively, β and n are parameters that can be determined by the values of $K(x)$, $N(x)$ and $\overline{m}(x)$ at $x = L_i/2$ and L_i or at the other control sections, L_i is the length of the ith step column.

The four linearly independent solutions of Eq. (2) for this case are found as

$$\left.\begin{aligned} S_{1i}(x) &= \eta_{1i} J_n(\eta_{1i}) \\ S_{2i}(x) &= \eta_{1i} Y_n(\eta_{1i}) \\ S_{3i}(x) &= \eta_{2i} I_n(\eta_{2i}) \\ S_{4i}(x) &= \eta_{2i} K_n(\eta_{2i}) \end{aligned}\right\} \quad n = \text{integer} \tag{5}$$

where $J_n(\cdot)$, $Y_n(\cdot)$, $I_n(\cdot)$ and $K_n(\cdot)$ are Bessel functions of the first, second, third and fourth kinds, respectively, η_{1i} and η_{2i} are given by

$$\left.\begin{aligned} &\eta_{1i} = \lambda_{1i}\sqrt{1+\beta_i \tfrac{x}{L_i}}, \quad \eta_{2i} = \lambda_{2i}\sqrt{1+\beta_i \tfrac{x}{L_i}}, \\ &\lambda_{1i} = \tfrac{2}{\beta}\sqrt{z_{1i}}, \quad z_{1i} = N_{ei} + \sqrt{N_{ei}^2 + \omega_{ei}^4} \\ &\lambda_{2i} = \tfrac{2}{\beta_i}\sqrt{z_{2i}}, \quad z_{2i} = N_{ei} - \sqrt{N_{ei}^2 + \omega_{ei}^4} \\ &N_{ei} = \tfrac{N_i(0)}{2K_i(0)}, \quad \omega_{ei}^4 = \tfrac{\overline{m}_i(0)\omega^2}{K_i(0)} \end{aligned}\right\} \tag{6}$$

Case 2. $K_i(x)$, $N_i(x)$ and $\overline{m}_i(x)$ are given by

$$\left.\begin{aligned} K_i(x) &= K_i(0)\left(1+\beta_i \tfrac{x}{L_i}\right)^{n+4} \\ N_i(x) &= N_i(0)\left(1+\beta_i \tfrac{x}{L_i}\right)^{n+2} \\ \overline{m}_i(x) &= \overline{m}_i(0)\left(1+\beta_i \tfrac{x}{L_i}\right)^{n} \end{aligned}\right\} \tag{7}$$

The four linearly independent solutions of Eq. (2) for this case are

$$S_{ji} = \left(1+\beta_i \frac{x}{L_i}\right)^{\gamma_{ji}} \quad j = 1,2,3,4 \tag{8}$$

where

$$\left.\begin{aligned} &\gamma_{1i} = -\tfrac{n+1}{2} + \sqrt{\tfrac{(n+1)^2}{4} + \omega_{fi} - N_{fi}} \\ &\gamma_{2i} = -\tfrac{n+1}{2} - \sqrt{\tfrac{(n+1)^2}{4} + \omega_{fi} - N_{fi}} \\ &\gamma_{3i} = -\tfrac{n+1}{2} + \sqrt{\tfrac{(n+1)^2}{4} - \omega_{fi} - N_{fi}} \\ &\gamma_{4i} = -\tfrac{n+1}{2} - \sqrt{\tfrac{(n+1)^2}{4} - \omega_{fi} - N_{fi}} \\ &\omega_{fi} = \omega_{di}^4 + N_{fi}^2, \quad \omega_{di} = \tfrac{\overline{m}_i(0)\omega^2 L_i^4}{\beta_i^4 K_i(0)} \\ &N_{fi} = \tfrac{1}{2}(N_{di} - n - 2), \quad N_{di} = \tfrac{N_i(0)L_i^2}{\beta_i^2 K_i(0)} \end{aligned}\right\} \tag{9}$$

The solutions expressed by Eq. (8) are valid for the case that γ_{ji} $(j = 1, 2, 3, 4)$ are real value roots only. It can be seen from Eq. (9) that since $\omega_{fi} > N_f$, γ_1 and γ_2 are real value roots. If γ_{3i} and γ_{4i} are complex values, $S_{3i}(x)$ and $S_{4i}(x)$ should be written as

$$S_{3i}(x) = \left(1 + \beta_i \frac{x}{L_i}\right)^{-(n+1)/2} \sin \sqrt{\omega_{fi} + N_{fi} - \frac{(n+1)^2}{4}} \ln \left(1 + \beta_i \frac{x}{L_i}\right) \tag{10}$$

$$S_{4i}(x) = \left(1 + \beta_i \frac{x}{L_i}\right)^{-(n+1)/2} \cos \sqrt{\omega_{fi} + N_{fi} - \frac{(n+1)^2}{4}} \ln \left(1 + \beta_i \frac{x}{L_i}\right) \tag{11}$$

The four linearly independent solutions of Eq. (1) for this case can be written in the form

$$X_i(\xi_i) = \sum_{j=1}^{4} C_{ji} \exp(\gamma_{ji} \xi_i) \tag{12}$$

where

$$\begin{aligned}
&\xi_i = \ln \left(1 + \beta_i \frac{x}{L_i}\right) \\
&\gamma_{1i,2i,3i,4i} = -\frac{1}{2}\left(n_i + 4 \pm \sqrt{f_i}\right) \pm \sqrt{\frac{1}{4}\left(n_i + 4 \pm \sqrt{f_i}\right)^2 - \frac{1}{2}\left(y_{1i} \pm \sqrt{y_{1i}^2 - 4e_i}\right)} \\
&e_{1i} = -\frac{\alpha_{1i}^2}{\beta_i^4}, \quad \alpha_{1i}^2 = \frac{\overline{m}_i(0)\omega^2 L_i^4}{K_i(0)}, \quad f_i = n + 4 + y_{1i} - \frac{\alpha_{2i}}{\beta_i^2}, \quad \alpha_{2i} = \frac{N_i(0) L_i^2}{K_i(0)} \\
&y_{1i} = \sqrt[3]{-\frac{q_i}{2} + \sqrt{\left(\frac{q_i}{2}\right)^2 + \left(\frac{p_i}{3}\right)^3}} + \sqrt[3]{-\frac{q_i}{2} - \sqrt{\left(\frac{q_i}{2}\right)^2 + \left(\frac{p_i}{3}\right)^3}}, \quad p_i = \frac{4\alpha_{1i}^2}{\beta_i^4} \\
&q_i = \frac{\alpha_{1i}^2}{\beta_i^4}\left[4\left(n + 4 - \frac{\alpha_{1i}}{\beta_i^2}\right) + \frac{4c_i}{3}\right] - \frac{2c_i^3}{27}, \quad c_i = (n+4)(n+3) + \frac{\alpha_{2i}}{\beta_i^2}
\end{aligned} \tag{13}$$

Case 3. The distributions of flexural stiffness, axial force and mass per unit length are described by the following exponential functions

$$\left.\begin{aligned}
K_i(x) &= K_i(0) \exp\left(b_i \tfrac{x}{L_i}\right) \\
N_i(x) &= N_i(0) \exp\left(b_i \tfrac{x}{L_i}\right) \\
\overline{m}_i(x) &= \overline{m}_i(0) \exp\left(b_i \tfrac{x}{L_i}\right)
\end{aligned}\right\} \tag{14}$$

The four linearly independent solutions of Eq. (2) for this case are derived as follows

$$\left.\begin{aligned}
S_{ji}(x) &= \exp\left(\gamma_{ji} \tfrac{x}{L_i}\right) \\
\gamma_{1i} &= \tfrac{b_i}{2} + \sqrt{\tfrac{b_i^2}{4} - z_{2i}} \\
\gamma_{2i} &= \tfrac{b_i}{2} - \sqrt{\tfrac{b_i^2}{4} - z_{2i}} \\
\gamma_{3i} &= \tfrac{b_i}{2} + \sqrt{\tfrac{b_i^2}{4} - z_{1i}} \\
\gamma_{4i} &= \tfrac{b_i}{2} - \sqrt{\tfrac{b_i^2}{4} - z_{1i}}
\end{aligned}\right\} \tag{15}$$

where z_{1i} and z_{2i} are given by Eq. (6).

It is evident that the above solutions are only valid for the case that γ_{ji} ($j = 1, 2, 3, 4$) are real value roots. Since $z_{2i} < 0$, γ_{1i} and γ_{2i} are real value roots. If $b_i^2/4 < z_{1i}$, γ_{3i} and γ_{4i} are complex value roots, for this case $S_{3i}(x)$ and $S_{4i}(x)$ should be determined by

$$S_{3i}(x) = \exp\left(\frac{b_i x}{2L_i}\right) \sin\left(\sqrt{z_{1i} - \frac{b_i^2}{4}}\frac{x}{L_i}\right) \tag{16}$$

$$S_{4i}(x) = \exp\left(\frac{b_i x}{2L_i}\right) \cos\left(\sqrt{z_{1i} - \frac{b_i^2}{4}}\frac{x}{L_i}\right) \tag{17}$$

The four linearly independent solutions of Eq. (1) for case 3 can also be expressed by Eq. (12), but the parameters involved should be determined by the following equations

$$\gamma_{1i,2i,3i,4i} = -\frac{1}{2}\left(\frac{b_i}{L_i} \pm \sqrt{y_{1i} - \frac{N_i(0)}{K_i(0)}}\right) \pm \sqrt{\frac{1}{4}\left(\frac{b_i}{L_i} \pm \sqrt{y_{1i} - \frac{N_i(0)}{K_i(0)}}\right)^2 - \frac{1}{2}\left(y_{1i} \pm \sqrt{y_{1i}^2 - 4e_i}\right)} \tag{18}$$

where $e_i = -(\overline{m}_i(0)\omega^2/K_i(0))$, y_{1i} is given by Eq. (13), but the parameters involved should be determined by

$$\left.\begin{aligned} q_i &= 4e_i\left[\tfrac{d_i}{3} - \tfrac{N_i(0)}{K_i(0)}\right] - \tfrac{2d_i^3}{27} \\ p_i &= 4e_i - \tfrac{d_i^3}{3}, \quad d_i = \left(\tfrac{b_i}{L_i}\right)^2 + \tfrac{N_i(0)}{K_i(0)} \end{aligned}\right\} \tag{19}$$

If $4b_i/L_i > 2d_i$, the sign before $\sqrt{y_{1i} - (N_i(0)/K_i(0))}$ should be the same as that before $\sqrt{y_{1i} - 4e_i}$, otherwise the signs must be different between each other.

Case 4. $\quad K_i(x) = K_i, \quad N_i(x) = N_i, \quad \overline{m}_i(x) = \overline{m}_i \qquad (20)$

This special case represents a stepped uniform column. The four linearly independent solutions of Eqs. (1) and (2), which are the same for this case, are found as

$$\left.\begin{aligned} S_{1i}(x) &= \sin k_{1i}x \\ S_{2i}(x) &= \cos k_{1i}x \\ S_{3i}(x) &= \sinh k_{2i}x \\ S_{4i}(x) &= \cosh k_{2i}x \end{aligned}\right\} \tag{21}$$

where

$$\left.\begin{aligned} k_{1i} &= \lambda_{1i}\sqrt{\sqrt{1 + \left(\tfrac{k_i}{\lambda_{1i}}\right)^4} + 1} \\ k_{2i} &= \lambda_{1i}\sqrt{\sqrt{1 + \left(\tfrac{k_i}{\lambda_{1i}}\right)^4} - 1} \\ \lambda_{1i}^2 &= \tfrac{N_i}{2K_i}, \quad k_i^2 = \tfrac{\overline{m}_i\omega^2}{K_i} \end{aligned}\right\} \tag{22}$$

If the distributions of flexural stiffness, axial force and mass per unit length of the ith step column do not obey the assumed expressions given in the above four cases, this step column should be divided into several segments such that the distributions of flexural stiffness, axial force and mass intensity in each of the segments may match accurately or approximately one of the expressions described in the four cases. The eigenvalue equation for the non-conservative stability of a multi-step non-uniform column can be conveniently established using the general solution, Eq. (3), the boundary conditions and the transfer matrix method to be introduced below. The FEM is usually considered as the most general approach for both static and dynamic analysis of structural systems. However, as pointed out by Sato (1980), the transfer matrix method has some advantages in computation: ease of programming, small memory requirements, and availability of ready-made transfer matrix catalogues for various elements. It will be shown through a numerical example in this paper that one of the advantages of the present transfer matrix method is that the total number of segments required could be much less than that normally needed in FEM analysis.

Using the general solution, Eq. (3), the mode shape functions of displacement $X_i(x)$, rotation $\theta_i(x)$, bending moment $M_i(x)$ and shear force $Q_i(x)$ can be expressed in a matrix form as follows

$$\begin{bmatrix} X_i(x) \\ \theta_i(x) \\ M_i(x) \\ Q_i(x) \end{bmatrix} = [S_i(x)] \begin{bmatrix} C_{1i} \\ C_{2i} \\ C_{3i} \\ C_{4i} \end{bmatrix} \tag{23}$$

where

$$[S_i(x)] = \begin{bmatrix} S_{1i}(x) & S_{2i}(x) & S_{3i}(x) & S_{4i}(x) \\ S'_{1i}(x) & S'_{2i}(x) & S'_{3i}(x) & S'_{4i}(x) \\ -K_i(x)S''_{1i}(x) & -K_i(x)S''_{2i}(x) & -K_i(x)S''_{3i}(x) & -K_i(x)S''_{4i}(x) \\ -[K_i(x)S_{1i}(x)]''' & -[K_i(x)S_{2i}(x)]''' & -[K_i(x)S_{3i}(x)]''' & -[K_i(x)S_{4i}(x)]''' \end{bmatrix} \tag{24}$$

The relation between the parameters X_{i1}, θ_{i1}, M_{i1} and Q_{i1} at the end $x = L_i$ and the parameters X_{i0}, θ_{i0}, M_{i0} and Q_{i0} at the end $x = 0$ of the ith step column can be expressed as

$$\begin{bmatrix} X_{i1} \\ \theta_{i1} \\ M_{i1} \\ Q_{i1} \end{bmatrix} = [T_i] \begin{bmatrix} X_{i0} \\ \theta_{i0} \\ M_{i0} \\ Q_{i0} \end{bmatrix} \tag{25a}$$

where

$$[T_i] = [S_i(x_{i1})][S_i(x_{i0})]^{-1} \tag{25b}$$

$$X_{i1} = X_i(L_i), \quad \theta_{i1} = \theta_i(L_i), \quad M_{i1} = M_i(L_i), \quad Q_{i1} = Q_i(L_i)$$

$$X_{i0} = X_i(0), \quad \theta_{i0} = \theta_i(0), \quad M_{i0} = M_i(0), \quad Q_{i0} = Q_i(0)$$

$[T_i]$ is called the transfer matrix because it transfers the parameters at the end $x = x_{i0}$ to those at the end $x = x_{i1}$. It is well known that the displacement, rotation, bending moment and shear force at the common interface of two neighbouring steps are required to be continuous, i.e.,

$$X_{i0} = X_{(i-1)1}, \quad \theta_{i0} = \theta_{(i-1)1}, \quad M_{i0} = M_{(i-1)1}, \quad Q_{i0} = Q_{(i-1)1} \tag{26}$$

Substituting Eq. (26) into Eq. (25a) leads to

$$\begin{bmatrix} X_{i1} \\ \theta_{i1} \\ M_{i1} \\ Q_{i1} \end{bmatrix} = [T_i] \begin{bmatrix} X_{(i-1)1} \\ \theta_{(i-1)1} \\ M_{(i-1)1} \\ Q_{(i-1)1} \end{bmatrix} \tag{27}$$

Using Eqs. (25a) and (27) yields

$$\begin{bmatrix} X_{i1} \\ \theta_{i1} \\ M_{i1} \\ Q_{i1} \end{bmatrix} = [T_i][T_{i-1}] \begin{bmatrix} X_{(i-1)0} \\ \theta_{(i-1)0} \\ M_{(i-1)0} \\ Q_{(i-1)0} \end{bmatrix} \tag{28}$$

The relation between the parameters X_{q1}, θ_{q1}, M_{q1} and Q_{q1} at the end of the last step column $x = L_q$ and the parameters X_{10}, θ_{10}, M_{10} and Q_{10} at the end of the first step column $x = 0$ can be established using Eqs. (25a) and (27) repeatedly as follows

$$\begin{bmatrix} X_{q1} \\ \theta_{q1} \\ M_{q1} \\ Q_{q1} \end{bmatrix} = [T] \begin{bmatrix} X_{10} \\ \theta_{10} \\ M_{10} \\ Q_{10} \end{bmatrix} \tag{29}$$

where

$$[T] = [T_q][T_{q-1}] \cdots [T_2][T_1] \tag{30}$$

and $[T]$ has the following form

$$[T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \tag{31}$$

The elements T_{ij} $(i, j = 1, 2, 3, 4)$ of $[T]$ can be determined from Eq. (30).

If there are a lumped mass, a rotational spring (with stiffness $K_{\varphi i}$) and a translational spring (with stiffness K_{ui}) attached at the interface of the $(i-1)$th step column and the ith step column, the displacement, rotation, bending moment and shear force are required to satisfy the following conditions (Fig. 2)

$$\left. \begin{aligned} X_{i0} &= X_{(i-1)1} \\ \theta_{i0} &= \theta_{(i-1)1} \\ M_{i0} &= M_{(i-1)1} - K_{\varphi i}\theta_{(i-1)1} \\ Q_{i0} &= Q_{(i-1)1} - (m_i\omega^2 - K_{ui})X_{(i-1)1} \end{aligned} \right\} \tag{32}$$

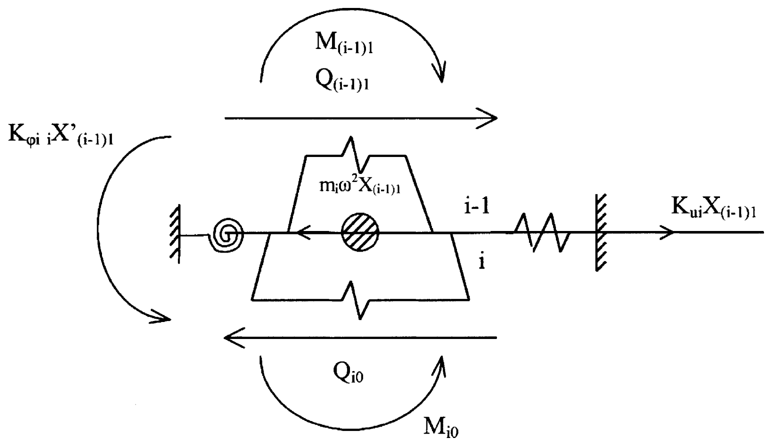

Fig. 2. The forces acting on the boundary between the $(i-1)$th step column and the ith step column.

For this case the transfer matrix $[T_i]$ should be replaced by

$$[T_{ism}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -K_{\varphi i} & 1 & 0 \\ -(m_i\omega^2 - K_{ui}) & 0 & 0 & 1 \end{bmatrix} [T_i] \tag{33}$$

The eigenvalue equation for the non-conservative stability of a multi-step non-uniform column can be established using Eq. (29) and the specific boundary conditions as follows

1. A multi-step cantilever column shown in Fig. 1.
   The boundary condition for this case are given by

$$M_{10} = 0, \quad Q_{10} = 0 \tag{34}$$

$$X_{q1} = 0, \quad \theta_{q1} = 0 \tag{35}$$

Applying the boundary conditions, Eq. (34), to Eq. (29) leads to

$$\begin{bmatrix} X_{q1} \\ \theta_{q1} \\ M_{q1} \\ Q_{q1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \begin{bmatrix} X_{10} \\ \theta_{10} \\ 0 \\ 0 \end{bmatrix} \tag{36}$$

The expressions of X_{q1} and θ_{q1} are obtained from Eq. (36) as follows

$$\left.\begin{aligned} X_{q1} &= T_{11}X_{10} + T_{12}X_{10} \\ \theta_{q1} &= T_{21}X_{10} + T_{22}\theta_{10} \end{aligned}\right\} \tag{37}$$

Applying the boundary conditions, Eq. (35), to Eq. (37) and considering the conditions, $X_{10} \neq 0$, $\theta_{10} \neq 0$, one obtains

$$T_{11}T_{22} - T_{12}T_{21} = 0 \tag{38}$$

This is the eigenvalue equation for the non-conservative stability of a multi-step cantilever column. In Eq. (38), T_{ij} $(i, j = 1, 2)$ can be determined from Eq. (30).

If a concentrated mass, m_1, attached at the top of the multi-step column, for this case the eigenvalue equation is

$$(T_{11} - m_1\omega^2 T_{14})T_{12} - (T_{21} - m_1\omega^2 T_{24})T_{22} = 0 \tag{39}$$

2. A multi-step column with concentrated masses and spring supports shown in Fig. 3.
   The boundary conditions for this case are given by

$$\left.\begin{aligned} M_{10} &= -K_{\varphi 1}\theta_{10} \\ Q_{10} &= -(m_1\omega^2 - K_{u1})X_{10} \end{aligned}\right\} \tag{40}$$

$$\left.\begin{aligned} M_{q1} &= K_{\varphi L}\theta_{q1} \\ Q_{q1} &= (m_L\omega^2 - K_{uL})X_{q1} \end{aligned}\right\} \tag{41}$$

Substituting Eqs. (40) and (41) into Eq. (29) leads to

$$\begin{bmatrix} X_{q1} \\ \theta_{q1} \\ -K_{\varphi L}\theta_{q1} \\ (m_L\omega^2 - K_{uL})X_{q1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \begin{bmatrix} X_{10} \\ \theta_{10} \\ -K_{\varphi 1}\theta_{10} \\ -(m_1\omega^2 - K_{u1})X_{10} \end{bmatrix} \tag{42}$$

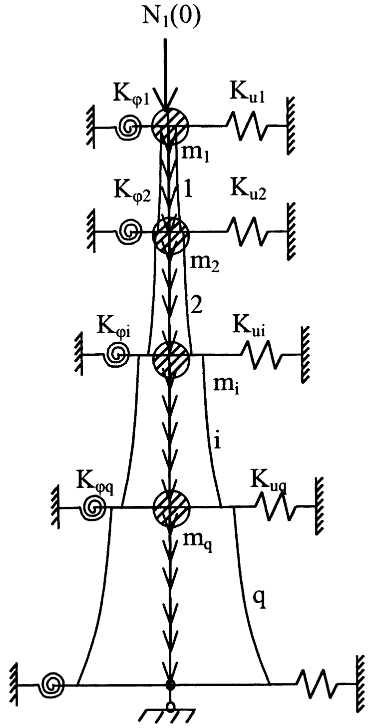

Fig. 3. A multi-step column with concentrated masses and spring supports.

The eigenvalue equation can be established from Eq. (42) as follows

$$A_{11}A_{22} - A_{12}A_{21} = 0 \tag{43}$$

where

$$\left.\begin{aligned} A_{11} &= (m_L\omega^2 - K_{uL})[T_{11} - (m_1\omega^2 - K_{u1})T_{14}] - [T_{41} - (m_1\omega^2 - K_{u1})T_{44}] \\ A_{12} &= (m_L\omega^2 - K_{uL})[T_{12} - K_{\varphi 1}T_{13}] - (T_{42} - K_{\varphi 1}T_{43}) \\ A_{21} &= K_{\varphi L}[T_{21} - (m_1\omega^2 - K_{u1})T_{24}] - [T_{31} - (m_1\omega^2 - K_{u1})T_{34}] \\ A_{22} &= K_{\varphi L}(T_{22} - K_{\varphi 1}T_{23}) - (T_{32} - K_{\varphi 1}T_{33}) \end{aligned}\right\} \tag{44}$$

T_{ij} $(i,j = 1,2,3,4)$ are the elements of $[T]$

$$[T] = [T_{qsm}][T_{(q-1)sm}]\cdots[T_{1sm}] \tag{45}$$

$[T_{ism}]$ is given by Eq. (33).

The eigenvalue equations for other boundary conditions can be obtained similarly using the aforementioned method.

3. Numerical example

A four-step cantilever column subjected to concentrated forces and variably distributed forces, as shown in Fig. 1, is considered here to illustrate the application of the proposed method. The flexural stiffness, axial force and mass per unit length of the ith step column are given by

$$K_i(x) = K_i(0)\left(1 + \beta_i \tfrac{x}{L_i}\right)^5$$
$$N_i(x) = N_i(0)\left(1 + \beta_i \tfrac{x}{L_i}\right)^3 \quad (i = 1, 2, 3, 4)$$
$$\overline{m}_i(x) = \overline{m}_i(0)\left(1 + \beta_i \tfrac{x}{L_i}\right)$$

where

$$L_1 = L_2 = L_3 = L_4 = L/4$$

$$\beta_1 = 0, \quad \beta_2 = 0.05, \quad \beta_3 = 0.1, \quad \beta_4 = 0.1$$

$$K_2(0) = 1.2K_1(0), \quad K_3(0) = 1.8K_1(0), \quad K_4(0) = 3.5K_1(0)$$

$$N_2(0) = 1.2N_1(0), \quad N_3(0) = 1.5N_1(0), \quad N_4(0) = 2.5N_1(0)$$

$$\overline{m}_2(0) = 1.2\overline{m}_1(0), \quad \overline{m}_3(0) = 1.5\overline{m}_1(0), \quad \overline{m}_4(0) = 2.0\overline{m}_1(0)$$

The procedure for determining the critical buckling force is as follows

1. Determination of the four linearly independent solutions.
   If the concentrated forces are follower loads, and the variably distributed forces are conservative loads, the solutions, $S_i(x)$ $(i = 1, 2, 3, 4)$, of the ith step column are given by Eq. (21) when $i = 1$, and Eq. (8) when $i \neq 1$, respectively.
   If all the concentrated forces and variably distributed forces are follower loads, the solutions, $S_i(x)$ $(i = 1, 2, 3, 4)$, of the ith step column are given by Eq. (21) when $i = 1$, and Eq. (12) when $i \neq 1$, respectively.
2. Determination of the transfer matrix.
   Using the four linearly independent solutions $S_i(x)$ $(i = 1, 2, 3, 4)$ and Eqs. (24) and (25b) leads to $[T_1]$. Then the total transfer matrix $[T]$ can be determined from Eq. (30).
3. Determination of the critical buckling force.

The boundary conditions are given by Eqs. (34) and (35), and the eigenvalue equation for this case is Eq. (38). Solving the eigenvalue equation one obtains

$$N_{1,\mathrm{cr}}^{(1)}(0) = 49.0735\frac{K_1(0)}{L}, \quad \text{or} \quad N_{1,\mathrm{cr}}^{(1)}(0) = 8.7059\frac{K_4(L_4)}{L},$$

for the case that all the concentrated forces and variably distributed forces are follower loads and

$$N_{1,\mathrm{cr}}^{(2)}(0) = 45.4196\frac{K_1(0)}{L}, \quad \text{or} \quad N_{1,\mathrm{cr}}^{(2)}(0) = 8.0577\frac{K_4(L_4)}{L},$$

for the case that the concentrated forces are follower loads, but the variably distributed forces are conservative loads.

The FEM with cubic approximation of displacements is also adopted to compare the results obtained by the proposed method. The column is divided into 40 uniform elements for the stability analysis. It is found:

$$N_{1,\mathrm{crf}}^{(1)}(0) = 49.0733\frac{K_1(0)}{L}, \quad \text{or} \quad N_{1,\mathrm{crf}}^{(1)}(0) = 8.7059\frac{K_4(L_4)}{L},$$

and

$$N_{1,\mathrm{crf}}^{(2)}(0) = 45.4194\frac{K_1(0)}{L}, \quad \text{or} \quad N_{1,\mathrm{crf}}^{(2)}(0) = 8.0577\frac{K_4(L_4)}{L},$$

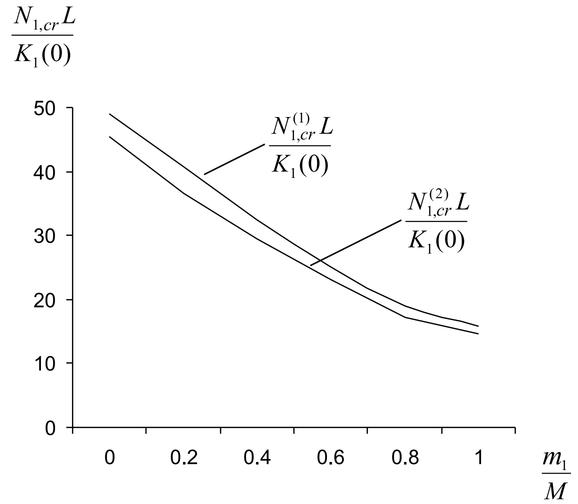

Fig. 4. The influence of the end concentrated mass on the critical force. Note: M is the total mass of the multi-step column.

It is evident that the results obtained from the proposed method and FEM are in close agreement. However, the number of segments used in the present transfer matrix method is much less than that used in FEM analysis. It is revealed from our computation that the proposed method takes less computational time than FEM, thus illustrating the present method is efficient, convenient and accurate.

If a concentrated mass m_1 is attached at the top of the multi-step column, the calculated critical forces for the two types of follower force system all decrease as the ratio of m_1 to M increases, where M is the total mass of the multi-step column. The influence of the end concentrated mass on the critical force is shown in Fig. 4. It can be seen from Fig. 4 that the effect of the end concentrated mass on the critical force is significant.

4. Conclusions

The governing differential equations for the non-conservative elastic stability of a non-uniform column subjected to (i) concentrated follower forces and variably distributed follower forces, and (ii) concentrated follower forces and variably distributed conservative forces, are established. The exact solutions of the governing equations for the non-conservative stability of four kinds of one-step non-uniform columns are derived for the first time. A new exact approach, which combines the exact solutions of one-step columns and the transfer matrix method, is presented for the non-conservative stability analysis of multi-step non-uniform columns. The advantage of the proposed method is that the eigenvalue equation for a multi-step non-uniform column with any kinds of two end support configurations, an arbitrary number of spring supports and concentrated masses can be conveniently determined from a second order determinant. As a consequence, the decrease in the determinant order, as compared with previously developed procedures, leads to significant savings in the computational effort. The numerical example shows that the results obtained from the proposed method are in good agreement with those determined from FEM, but the proposed method takes less computational time than FEM, thus illustrating the present procedure is an exact and efficient method. It is shown through the numerical example that the critical buckling force of a multi-step non-uniform column subjected to concentrated and variably distributed follower forces is greater than that of the column under concentrated follower forces and variably distributed conservative forces. The effect of the end concentrated mass on the critical force is found to be significant.

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